Unified statistical thermodynamics of quantum gases trapped under generic power law potential in d dimension and equivalence in $d = 1$

Mir Mehedi Faruk

Department of Theoretical Physics, University of Dhaka, Dhaka-1000
Theoretical Physics, Blackett Laboratory, Imperial College, London SW7 2AZ, United Kingdom
Email: muturza3.1416@gmail.com, mehedi.faruk.mir@cern.ch

June 18, 2015

Abstract

A unified description for the Bose and Fermi gases trapped in an external generic power law potential $U = \sum_{i=1}^{d} c_i |\frac{x_i}{a_i}|^{n_i}$ is presented using the grandpotential of the system in d dimensional space. The thermodynamic quantities of the quantum gases are derived from the grand potential. An equivalence between the trapped Bose and Fermi gases is constructed in one dimension ($d = 1$) using the Landen relation. It is also found that the established equivalence between the ideal free Bose and Fermi gases in $d = 2$ (M. H. Lee, Phys. Rev. E 55, 1518 (1997)) is lost when external potential is applied.

1 Introduction

The two types of quantum gases manifest different thermodynamic behaviour due to inherent difference of their statistical distribution[1, 2]. Fermi gas, which obeys Pauli exclusion principle exhibit distinct characteristic such as zero point energy and pressure[1, 2] whether Bose gas condensates[1, 2], not obeying this principle. The thermodynamic properties of the Bose and Fermi gases are determined by Bose and Fermi function[1, 3] which have different mathematical structures. But an unified formulation for quantum gases was presented recently and Lee[4, 5, 6] established a remarkable equivalence between ideal free Bose and Fermi gases in $d = 2$. The equivalence is based on a certain invariance of the polylogarithms under Euler transformation[7] of the fugacities. After the inspiring work of May[8] considerable attactions are drawn to study further the equivalence between quantum gases. Point to note, plenty of study are made to investigate the thermodynamic properties of quantum gases under trapping potential[9, 10, 11, 12] after it was possible to create Bose-Einstein condensate in magnetically trapped Alkali gases[13, 14, 15]. It was demonstrated in recent papers that, trapping potential can change the characteristics of quantum gases. For instance, although there is no Bose condensation for ideal Bose gas in $d < 3$,[9, 16] it was found that in presence of trapping potential, Bose condensate can form in $d < 3$[9, 16]. So, it will be intruiging to check the status of this equivalence found by Lee[4] in case of trapped system.

A lot of efforts are made to understand dimensional dependence of different properties of quantum gases such as condensation[17], [18], conductivity[19], transport properties[20], degeneracy[24], polylogarithmis[6], q-deforemed syetm[25, 26]. Ib this report, at first the grand potential for quantum gases under generic power law potential is determined in d dimensional space. The thermodynamic quantities are then derived from the grand potential. From the general expressions of the calculated thermodynamic quantities we have investigated closely the case with $d = 1$ and $n_1 = 2$ (harmonic potential) and found an equivalence can be obtained between the thermodynamic quantities of Bose and Fermi gases. It is also seen, the established equivalence for ideal free quantum gases in $d = 2$ disappears when a external potential is applied.

The report is organized in the following way. The grand potential of quantum gases under generic power law potential is calculated in section 2. In section 3 we have presented the thermodynamic quatities in an unified way for both types of quantum gases. The useful landen relations are explored in section 4. And section 5 is devoted to explore the equivalence in $d = 1$ with harmonically trapped quantum gases. A discussion on the equivalence of free and trapped quantum gases is presented in section 6. The report is concluded in section 7.

2 Grand potential

Considering an ideal quantum system trapped in a generic power law potential in d dimensional space with a single particle Hamiltonian,

$$\epsilon(p, x_i) = bp^l + \sum_{i=1}^{d} c_i \left| \frac{x_i}{a_i} \right|^{n_i} \tag{1}$$

Where, b, l, a_i, c_i, n_i are all postive constants, p is the momentum and x_i is the i th component of coordinate of a particle. Here, c_i, a_i, n_i determines the depth and confinement power of the potential and l being the kinematic parameter. Now, the well known formula of density of states [9, 16],

$$\rho(\epsilon) = \int \int \frac{d^d r d^d p}{(2\pi\hbar)^d} \delta(\epsilon - \epsilon(p, r)) \tag{2}$$

So, from the above equation density of states is[9, 16],

$$\rho(\epsilon) = B \frac{\Gamma(\frac{d}{l} + 1)}{\Gamma(\chi)} \epsilon^{\chi - 1} \tag{3}$$

where,

$$B = \frac{V_d C_d}{h^d a^{d/l}} \prod_{i=1}^{d} \frac{\Gamma(\frac{1}{n_i} + 1)}{c_i^{\frac{1}{n_i}}} \tag{4}$$

Here, $C_d = \frac{\pi^{\frac{d}{2}}}{\Gamma(d/2+1)}$, $V_d = 2^d \prod_{i=1}^{d} a_i$ is the volume of an d-dimensional rectangular whose i-th side has length $2a_i$. $\Gamma(l) = \int_0^\infty dx x^{l-1} e^{-x}$ is the gamma function and $\chi = \frac{d}{l} + \sum_{i=1}^{d} \frac{1}{n_i}$.

The grand potential of quantum gases can be written as[1],

$$q = \frac{1}{a} \sum_{\epsilon} ln(1 + az exp(-\beta\epsilon)) \tag{5}$$

$\beta = \frac{1}{kT}$, where k being the Boltzmann Constant and $z = \exp(\beta\mu)$ is the fugacity, where μ being the chemical potential. a is equal to -1 for Fermi system and +1 for Bose system. In experiments with trapped gases, thermal energies far exceed the level spacing[14]. So, using the Thomas-Fermi semiclassical approximation[27] and re-writing the previous equation,

$$q = q_0 + \frac{1}{a} \int_0^\infty \ln(1 + az \exp(-\beta\epsilon)) \rho(\epsilon) d\epsilon \tag{6}$$

Here, $q_0 = \frac{1}{a} \ln(1 + az)$. Now finally the grand potential stands as,

$$q = \begin{cases} q_0 + B\Gamma(\frac{d}{l} + 1)(kT)^\chi f_{\chi+1}(z) & \text{,Fermi system} \\ q_0 + B\Gamma(\frac{d}{l} + 1)(kT)^\chi g_{\chi+1}(z) & \text{, Bose system} \end{cases} \tag{7}$$

Here, $g_l(z)$ and $f_l(z)$ are Bose and Fermi function respectively. Defined as

$$g_l(z) = \int_0^\infty \frac{x^{l-1}}{z^{-1}e^x - 1} = \sum_{j=1}^{\infty} \frac{z^l}{j^l} \tag{8}$$

$$f_l(z) = \int_0^\infty \frac{x^{l-1}}{z^{-1}e^x + 1} = \sum_{j=1}^{\infty} (-1)^j \frac{z^l}{j^l} \tag{9}$$

Now, Bose and Fermi functions can be written in terms of Polylogarithmic functions,

$$Li_q(t) = g_q(t) \tag{10}$$

$$Li_q(-t) = -f_q(t) \tag{11}$$

where, $Li_q(m)$ is the polylog of q and m. If $q \geq 1$, $Li_q(m)$ is analytic everywhere. It is a real valued function if $m \in \mathbb{R}$ and $-\infty < m < 1$. A useful integral representation of polylog is

$$Li_q(m) = \frac{1}{\Gamma(q)} \int_0^m [\ln(\frac{m}{\eta})]^{q-1} \frac{d\eta}{1 - \eta}, \tag{12}$$

for $Re(m) < 1$. To write the grand potential compactly, defining a quantity σ as,

$$\sigma = \begin{cases} -z & , \text{Fermi system} \\ z & , \text{Bose system} \end{cases} \tag{13}$$

So, re writing the grand potential,

$$q = q_0 + sgn(\sigma)B\Gamma(\frac{d}{l} + 1)(kT)^\chi Li_{\chi+1}(\sigma) \tag{14}$$

3   Statistical thermodynamics of trapped quantum gases

The number of particles N can be obtained,

$$N = z(\frac{\partial q}{\partial z})_{\beta, V}$$

$$\Rightarrow \quad N - N_0 = N_e = sgn(\sigma)\frac{V'_d}{\lambda'^\chi}Li_\chi(\sigma) \tag{15}$$

$$\Rightarrow \quad \rho = \frac{N_e}{V'_d} = sgn(\sigma)\frac{1}{\lambda'^d}Li_\chi(\sigma) \tag{16}$$

Where V'_d and λ' are defined as [16]

$$V'_d = V_d \prod_{i=1}^{d}(\frac{kT}{c_i})^{1/n_i}\Gamma(\frac{1}{n_i} + 1) \tag{17}$$

$$\lambda' = \frac{hb^{\frac{1}{l}}}{\pi^{\frac{1}{2}}(kT)^{\frac{1}{l}}}[\frac{d/2 + 1}{d/l + 1}]^{1/d} \tag{18}$$

It is noteworthy,

$$\lim_{n_i \to \infty} V'_d = V_d \tag{19}$$

$$\lim_{n_i \to \infty} \chi = \frac{d}{l} \tag{20}$$

$$\lim_{l \to 2, b \to \frac{1}{2m}} \lambda' = \lambda = \frac{h}{(2\pi mkT)^{1/2}} \tag{21}$$

Now, the other thermodynamic quantities in case of trapped system can be calculated from grand potential as below,

$$U = -(\frac{\partial q}{\partial \beta})_{z, V'_d} = NkT\chi\frac{Li_{\chi+1}(\sigma)}{Li_\chi(\sigma)} \tag{22}$$

$$S = kT(\frac{\partial q}{\partial T})_{z, V'_d} - Nk\ln z + kq = Nk(\chi + 1)\frac{Li_{\chi+1}(\sigma)}{Li_\chi(\sigma)} - \ln|\sigma| \tag{23}$$

$$P = \frac{1}{\beta}(\frac{\partial q}{\partial V'_d})_{\beta, z} = NkT\frac{1}{V'_d}\frac{Li_{\chi+1}(\sigma)}{Li_\chi(\sigma)} \tag{24}$$

$$C_V = T(\frac{\partial S}{\partial T})_{N, V'_d} = Nk[\chi(\chi + 1)\frac{Li_{\chi+1}(\sigma)}{Li_\chi(\sigma)} - \chi^2\frac{Li_\chi(\sigma)}{Li_{\chi-1}(\sigma)}] \tag{25}$$

$$\kappa_T = -V'_d(\frac{\partial V'_d}{\partial P'})_{N, T} = \frac{V'_d}{NkT}\frac{Li_{\chi-1}(\sigma)}{Li_\chi(\sigma)} \tag{26}$$

The above expressions compactly represent the thermodynamic quantities related to trapped Bose[16, 9] and Fermi gas[28, 29] In case of free system (all $n_i \longrightarrow \infty$) the above quantities reduce to,

$$\rho = sgn(\sigma)\frac{1}{\lambda^d}Li_{\frac{d}{2}}(\sigma) \tag{27}$$

$$U = NkT\frac{d}{l}\frac{Li_{\frac{d}{l}+1}(\sigma)}{Li_{\frac{d}{l}}(\sigma)} \tag{28}$$

$$S = Nk(\frac{d}{l}+1)\frac{Li_{\frac{d}{l}+1}(\sigma)}{Li_{\frac{d}{l}}(\sigma)} - \log|\sigma| \tag{29}$$

$$P = NkT\frac{1}{V_d}\frac{Li_{\frac{d}{l}+1}(\sigma)}{Li_{\frac{d}{l}}(\sigma)} \tag{30}$$

$$C_V = Nk[\frac{d}{l}(\frac{d}{l}+1)\frac{Li_{\frac{d}{l}+1}(\sigma)}{Li_{\frac{d}{l}}(\sigma)} - (\frac{d}{l})^2\frac{Li_{\frac{d}{l}}(\sigma)}{Li_{\frac{d}{l}-1}(\sigma)}] \tag{31}$$

$$\kappa_T = \frac{V_d}{NkT}\frac{Li_{\frac{d}{l}-1}(\sigma)}{Li_{\frac{d}{l}}(\sigma)} \tag{32}$$

So, choosing $l = 2$ in case of non-relativistic quantum gas, the Eq. (27)-(32) reduces to those in Ref.[3] for arbitrary dimension. And with $d = 3$, they reproduce the thermodynamic quantities for free Bose and Fermi gas[1, 2].

4 Landen Relation

The unified formulation shows that the thermodynamic quantities are described by the structural properties of polylogs. Landen[7] found relation between monolog and dilog, which is the key to make the equivalence between ideal free quantum gases[4] as well as trapped gases. If x_1 is a real number and $x_1 < 1$ and there exist a variable x_2, such that,

$$x_2 = -\frac{x_1}{1 - x_1} \tag{33}$$

then one finds,

$$Li_0(x_1) = -\frac{Li_0(x_2)}{Li_0(x_2) + 1} \tag{34}$$

$$Li_1(x_1) = -Li_1(x_2) \tag{35}$$

$$Li_2(x_1) = -Li_2(x_2) - \frac{1}{2}[Li(x_2)]^2 \tag{36}$$

The proof of the above relations are included in Appendix of Ref. [4]. These relations indicate Euler transformation[4] of x_1 to x_2.

5 Application in $d = 1$ for trapped gas

Note, in both case of free and trapped system the thermodynamic quantities are described by polylogs $Li_m(z)$. Now the polylogs are related to each other by landen relations, and the respective variables are related to each other by Euler transformation. In free system the polylogs describing the thermodynamic system are functions of dimension, while in of trapped system the polylogs describing the thermodynamic system are function of dimension, fugacity and power law exponents. In trapped system the dependence of polylogs on dimension and power law exponents are described by $\chi = \frac{d}{l} + \sum_{i=1}^{d}\frac{1}{n_i}$.

As $l = 2$, in case of nonrelativistic massive Boson and choosing $d = 1$, $n_1 = 2$ (harmonic potential),

$$\chi = \frac{1}{2} + \frac{1}{2} = 1 \tag{37}$$

If the densities are made the same, turning our attention towards, density ρ, with $\chi = 1$, we get from Eq. (27)

$$\rho\lambda = Li_1(z_B) = -Li_1(-z_F) \tag{38}$$

4

where, z_B and z_F denotes fugacity of Bose and Fermi gas respectively. So, according to Eq. (33) they are related to each other by Euler transformation. So, we can write following relation

$$z_F = \frac{z_B}{1 - z_B} \tag{39}$$

So, the fugacities are related to each other by Euler transformation, if we put $z_B = x_1$ and $z_F = -x_2$. Then, we can easily use the thermodynamic quantities to establish the equivalence. First turning our attention towards internal energy $U(z_B)$ with $d = 1$ and $n = 2$,

$$
\begin{aligned}
U(z_B) &= NkT\frac{Li_2(z_B)}{Li_1(z_B)} = NkT\frac{Li_2(x_1)}{Li_1(x_1)} = NkT\frac{-Li_2(x_2) - \frac{1}{2}(Li(x_2))^2}{-Li_1(x_2)} = NkT[\frac{Li_2(x_2)}{Li_1(x_2)} + \frac{1}{2}Li_1(x_2)] \\
&= U(z_F) + NkT\frac{1}{2}Li_1(-z_F) \\
&= U(z_F) + NkT\rho\lambda
\end{aligned}
$$

Now point to note, $\rho = \frac{N_c}{V'_d}$ and $V'_d \propto \sqrt{T}$. Also $\lambda \propto \frac{1}{\sqrt{T}}$ So, obviously the second term is temperature independent. As it turns out, the second term exactly corresponds to ground state energy [28] just in the case of ideal free quantum gases[4]. Hence, it can be concluded if the two reduced densities are the same, the fugacities are related by Euler transformation and as a result internal energies of Bose and Fermi gases only differ by the ground state energy of the Fermi gas only. So, denoting ground state energy by U_0, we can rewrite,

$$U(z_B) = U(z_F) - U_0 \tag{40}$$

Since, pressure and energy are related by $PV'_d = \frac{E}{\chi}$, from the help of Eq. (40), one can get

$$P(z_B) = P(z_F) - P_0 \tag{41}$$

Where, P_0 denotes ground state pressure of Fermi gas[16]. Now turning our attention towards entropy,

$$
\begin{aligned}
S(z_B) &= Nk[2\frac{Li_2(z_B)}{Li_1(z_B)} - log(z_B)] = Nk[2\frac{Li_2(x_1)}{Li_1(x_1)} - log(x_1)] = Nk[2\frac{Li_2(x_2) + \frac{1}{2}[Li_1(x_2)]^2}{Li_1(x_2)} - log(\frac{-x_2}{1 + x_2})] \\
&= Nk[2\frac{Li_2(x_2)}{Li_1(x_2)} + Li_1(x_2) - log(-x_2) + log(1 + x_2)] = Nk[2\frac{Li_2(x_2)}{Li_1(x_2)} - log(-x_2)] \\
&= Nk[2\frac{Li_2(-z_F)}{Li_1(-z_F)} - log(z_F)] = S(z_F) \tag{42}
\end{aligned}
$$

Here, we have used the identity $log(1 + x) = -Li_1(x)$. Also, it is clear that entropy remain exactly same for two types of quantum gases in this case. Now, from the equation of specific heat,

$$
\begin{aligned}
C_V(z_B) &= Nk[2\frac{Li_2(z_B)}{Li_1(z_B)} - \frac{Li_1(z_B)}{Li_0(z_B)}] \\
&= Nk[2\frac{Li_2(x_1)}{Li_1(x_1)} - \frac{Li_1(x_1)}{Li_0(x_1)}] = Nk[2\frac{Li_2(x_1)}{Li_1(x_1)} - \frac{Li_1(x_1)}{Li_0(x_1)}] \\
&= Nk[2\frac{Li_2(x_2) + \frac{1}{2}[Li_1(x_2)]^2}{Li_1(x_2)} - \frac{Li_1(x_2)[1 + Li_0(x_2)]}{Li_0(x_2)}] \\
&= Nk[2\frac{Li_2(x_2)}{Li_1(x_2)} - \frac{Li_1(x_2)}{Li_0(x_2)}] = Nk[2\frac{Li_2(-z_F)}{Li_1(-z_F)} - \frac{Li_1(-z_F)}{Li_0(-z_F)}] = C_V(z_F) \tag{43}
\end{aligned}
$$

This type of result is previously found by May[8] for free quantum gases in two dimensional space. In case of the isothermal compressibilty,

$$\kappa_T(z_B) = \frac{V'_d}{NkT}\frac{Li_0(z_B)}{Li_1(z_B)} = (1 + z_F)\kappa_T(z_F) \tag{44}$$

So, the isothermal compressibilty are not equivalent at all temperatures. If $z_F \longrightarrow \infty$ (i.e $z_B \longrightarrow 1$), $\kappa_T(z_F) \propto \frac{1}{log(z_F)}$ but $\kappa_T(z_B) \propto \frac{z_F}{log(z_F)}$. The latter diverges while the former vanishes. But if $z_F \longrightarrow 0$ (in the classical limit) the two of

course become equivalent. The same conclusion applies to the number fluctuation.

Finally, the reasons behind this remarkable equivalence can be found from the grand potential Q. Rewriting Q,

$$
\begin{aligned}
\log Q(z_B) &= \frac{V'_d}{\lambda} Li_2(x_B) = \frac{V'_d}{\lambda} Li_2(x_1) = \frac{V'_d}{\lambda}\left(-Li_2(x_2) - \frac{1}{2}[Li_1(x_2)]^2\right) \\
&= \frac{V'_d}{\lambda}\left(-Li_2(-z_F) - \frac{1}{2}[Li_1(z_F)]^2\right) \\
&= \log Q(z_F) - \frac{V'_d}{2}\rho^2\lambda^3
\end{aligned}
\tag{45}
$$

With careful inspection it can be seen from Eq. (16) - (18), the second term ib Eq. (45) is linear in β. So, the grand partition function of the two systems are related to each other by a term linear in β. As all the thermodynamic quantities are basically dervied from grand potential, thus we are able to make such connection for all the thermodynamic quantities. So, when we take first derivative of grand potential with respect to β, the obtained thermodynamic quantity internal energy of Bose and Fermi system only differ by a constant (*i.e.* the ground state energy) which is independent of β. And when we take the second derivative of grand potential with respect to β, the derived thermodynamic quantity specific heat are equal to each other.

6   Discussion

In this paper, we have seen once again if the fugacities of Bose and Fermi gas are related by Euler transformation an equivalence relation can be establised between the two types of quantum gases. One can check the status of the equivalence relation in $d = 2$ [4] for trapped quantum gases. Now re-writing the equation of reduced density (Eq. 27) with $l = 2$,

$$
\rho\lambda^2 = Li_{\frac{d}{2}+\sum_i \frac{1}{n_i}}(z_B) = -Li_{\frac{d}{2}+\sum_i \frac{1}{n_i}}(-z_F)
\tag{46}
$$

Now, choosing $d = 2$, the above expression reduces to,

$$
\rho\lambda^2 = Li_{1+\sum_{i=1,2} \frac{1}{n_i}}(z_B) = -Li_{1+\sum_{i=1,2} \frac{1}{n_i}}(-z_F)
$$

From Eq. (35) one can see the Euler transformation type relation between fugacities are possible only for monologs. So, as it stands from the above equation the Euler transformation type relation between fugacities are possible if and only if $\sum_{i=1,2} \frac{1}{n_i} = 0$. Now as, $n_1, n_2 > 0$ this criterion is possible if and only if $n_1 \longrightarrow \infty$ and $n_2 \longrightarrow \infty$, which is basically the condition for free system [9, 12, 16, 28]. So, the equivalence relation between the quantum gases is possible in two dimensional space only for the free system. This phenomenon is due to the fact, that trapped Bose gas actually condensates in $d = 2$ with any trapping potential or in more general, BEC is possible if and only if $\chi > 1$[9, 12, 16]. So, the equivalence relation between the Bose and Fermi gases are possible where BEC could not take place. Now in case of $d = 1$, Eq. (46) becomes,

$$
\rho\lambda = Li_{\frac{1}{2}+\frac{1}{n_1}}(z_B) = -Li_{\frac{1}{2}+\frac{1}{n_1}}(-z_F)
\tag{47}
$$

Again Eq. (35) suggests Euler transformation type relation between fugacities are possible in $d = 1$ if and only if quantum gases are trapped in harmonic potential ($n_1 = 2$).

7   Conclusion

From the unified statistical thermodynamics of quantum gases trapped under generic power law potential in d dimension, a case is shown with $d = 1$ where, Bose and Fermi gases can be treated as equivalent. This is possible only when the quantum gas is trapped in harmonic potential. It will be interesting to check the effect of interaction on this equivalence as well as to do the whole calculation with relativistic hamiltonian.

8   Acknowledgement

References

[1] R. K. Pathria, *Statistical Mechanics*, Elsevier, 2004.

[2] K. Huang, *Statistical Mechanics*, Wiley Eastern Limited, 1991.

[3] R. M. Ziff, G. E Uhlenbeck, M. Kac, phys. Reports 32 169 (1977).

[4] M. Howard Lee, Phys. Rev. E, 55 2 (1997)

[5] M. H. Lee, J. Math. Phys. 36, 1217 (1995).

[6] M. Howard Lee, Acta Phys. Polonica, 40 5 (2009)

[7] M. Howard Lee, Phys. Rev. E, 56 4 (1997)

[8] R. M. May, Phys. Rev. 135, A1515 1964.

[9] Luca Salasnich, J. Math. Phys 41, 8016 (2000).

[10] F. Dalfovo, S. Giorgini, L.P. Pitaevskii, and S. Stringari, Rev. Mod. Phys. 71 (1999)

[11] L. Salasnich, A. Parola, and L. Reatto, Phys. Rev. A 59, 2990 (1999)

[12] A. Jellal and M. Doud Mod. Phys Lett. B 17 1321 (2003).

[13] C. C. Bradley, C. A. Sackett, J. J. Tollett and R. G. Hulet, Phys. Rev. Lett. 75, 1687, (1995).

[14] M. H. Anderson, J. R. Esher, M. R. Mathews, C. E. Wieman and E. A. Cornell, Science 269, 195, (1995).

[15] K. B. Davis, M. O. Mewes, M. R. Andrew, N. J. Van Druten, D. S. Durfee, D. M. Kurn and W. Ketterle, Phys. Rev. Lett., 1687, 75, 1995.

[16] Mir Mehedi Faruk, arXiv:1502.07054

[17] A Bhowal, M Acharyya, Acta Phys. Polonica B, 43, 9 (2012).

[18] M. Acharyya, Commun. Theor. Phys. 55, 901 (2011).

[19] M. Acharyya, Commun. Theor. Phys. 56, 943 (2011).

[20] Mario Collura and Gabriele Martelloni, J. Stat. Mech. (2014) P08006.

[21] M. Howard Lee, Phys. Rev. E, 55, (1997).

[22] R. K. Pathria, Phys. Rev. E, 57 (1998).

[23] M. Apostol, Phys. Rev. E, 56, (1997).

[24] B. DeMarco and D. S. Jin, Science, 285, (1999).

[25] Shukuan Cai, Guozhen Su and Jincan Chen, . Phys. A: Math. Theor. 40 11245 (2007).

[26] P. Narayana Swamy, Eur. Phys. J. B 50, 291–294 (2006)

[27] T. T. Chou, C. N. Yang and L. H. Yu, Phys. Rev. A 53, 4257 (1996).

[28] Mir Mehedi Fauk, arXiv:1504.06050

[29] Mingzhe Li, Zijun Yan, Jincan Chen, Lixuan Chen and Chuanhong Chen, Phys Rev. A, 58 (1998).